INTEGRABILITY STRUCTURES IN THE RELATIVISTIC TWO-BODY PROBLEM

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CONTENTS

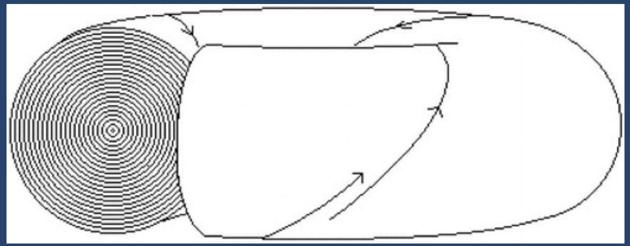
HAMILTONIAN INTEGRABLE DYNAMICS

PROMISE OF INTEGRABILITY

- A smooth, analytically understandable structure
- Multi-timescale expansions
- Separability of equations of motion
- Closed-form solutions of evolution
- No deterministic chaos – errors grow only polynomially in time

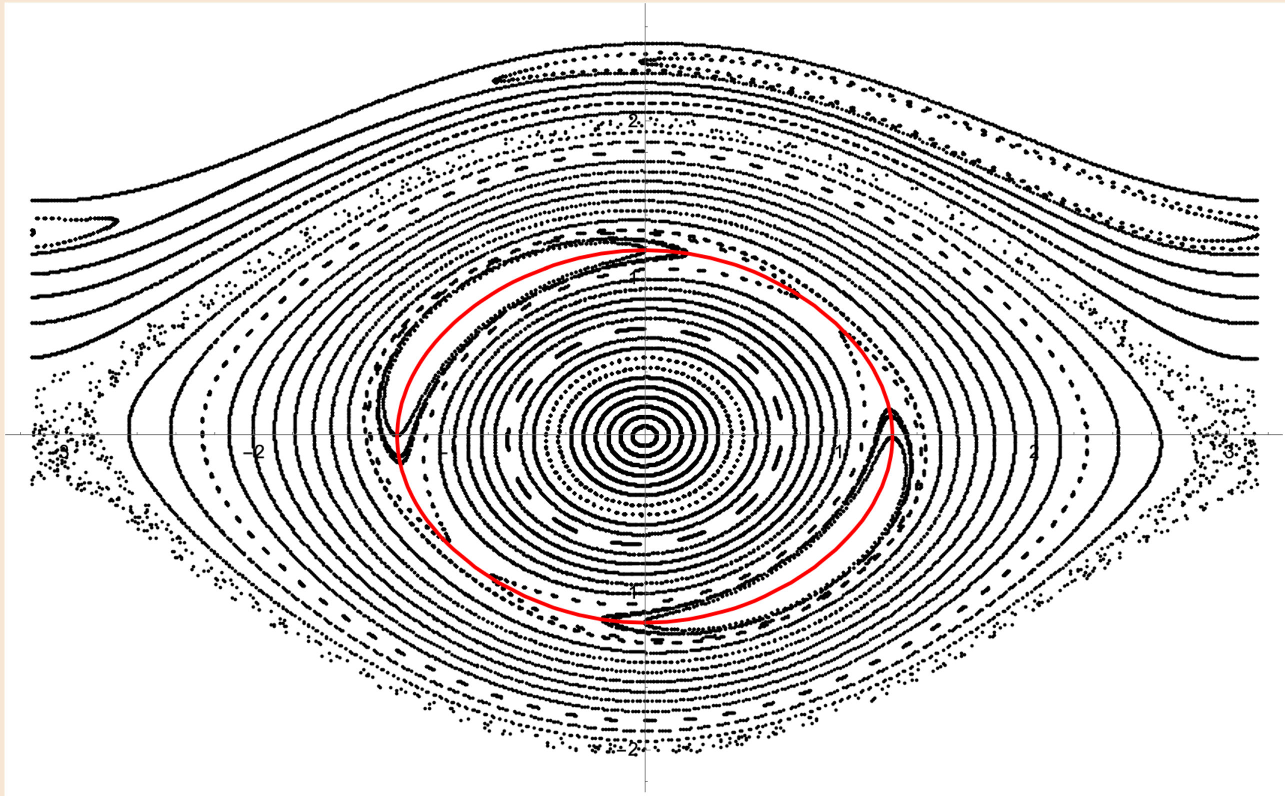
An autonomous Hamiltonian system of N degrees of freedom Liouville integrable if N functionally independent constants of motion in involution $F_1, F_2, \dots F_N$. Then:

- Evolution can be solved for by *quadratures*.
- ∃ pairs of canonical coordinates J_i, ψ^i such that J_i constants of motion, the Hamiltonian becomes H(J_i), and
  $$\dot{\psi}^i = \frac{\partial H}{\partial J_i} \equiv \Omega^i(J_j) = const.$$
  These are known as *Action-Angle coordinates*.



KOLMOGOROV-ARNOLD-MOSER THEOREM

- Given an integrable Hamiltonian $H_0(J_i)$, how crazy can the dynamics generated by a close perturbed Hamiltonian $H = H_0(J_i) + \epsilon H_1\ (J_i, \psi^j)$ get?
- If the frequencies are non-degenerate (non-constant and functionally independent on J_i), then most of the invariant tori survive and smoothly deformed.
- However, an $O(\sqrt{\epsilon})$ volume of tori where the frequencies are commensurate (resonant) $k_i\Omega^i = 0$ is replaced by *qualitatively different* structures
- Pragmatically:
  - Width of resonance scales with k_{i}-harmonic of perturbing Hamiltonian as $\sqrt{\epsilon H_{1k}}$.
  - For smooth functions harmonics go as $\mathrm{e}^{-C|k|}$ for large k_{i}.
  - For finite ϵ, we can set a $\propto \epsilon^2$ cut-off of the size of resonances to worry about – only finitely many



CONSERVATIVE APPROXIMATIONS TO THE TWO-BODY PROBLEM

INTEGRALS FROM POINCARÉ SYMMETRIES

- The Hamiltonian of a closed relativistic system must commute with generators of Poincaré group, this implies constants of motion
  - Spatial translations – total momentum $\vec{P}$
  - Time translations – total energy E (coordinate-time Hamiltonian)
  - Boosts – center of mass motion $\vec{K}$ (allows to decouple total momentum from relative momentum)
  - Rotations – total angular momentum $\vec{J}$
- Not all 10 commute, but we get ***6 commuting integrals***
  - Energy E,
  - Total momentum $\vec{P}$,
  - Magnitude and one component of angular momentum in COM frame $\vec{J}_{COM}$ (Pauli-Lubanski vector)

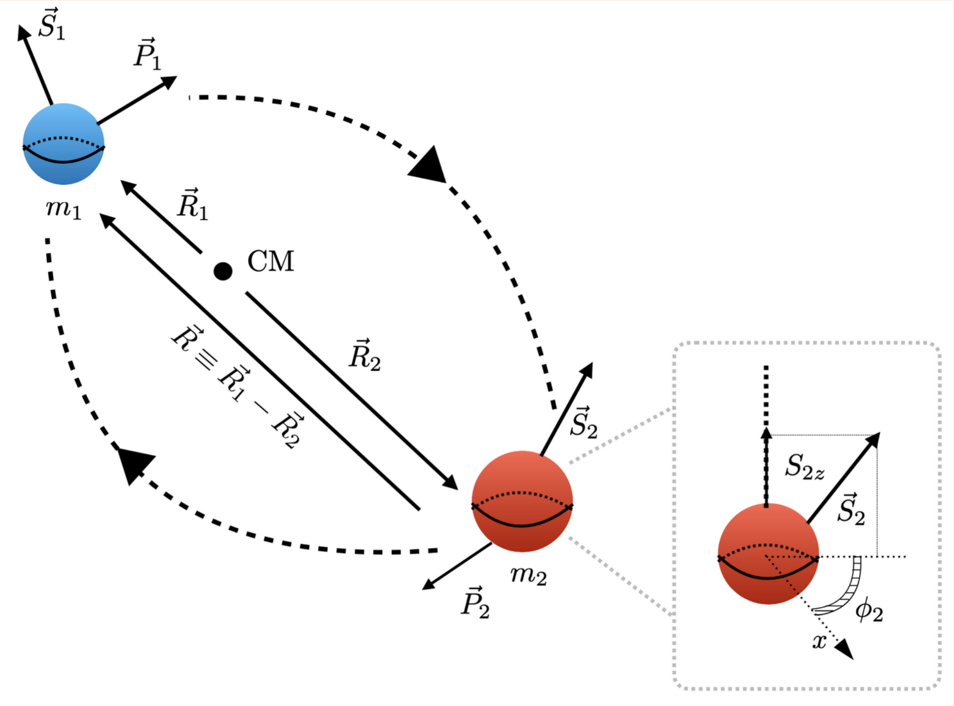
2 point particles
=
6 degrees of freedom
=
Automatically integrable

Any more degrees of freedom ”activate”?

- Integrability is not guaranteed

SPIN HAS ENTERED THE CHAT

- Spin-orbital coupling kicks in at 1.5PN or for rapidly spinning compact objects
- The system contains at least one more degree of freedom per particle – *orientation of the spin vector with respect to other characteristic vectors of the system*
- This picture applies to both PN and large-mass ratio limits
- *Will the system stay integrable?* ***No fundamental symmetries guarantee it***



From Tanay+ (2012.06586)

KNOWN INTEGRABILITY RESULTS

Approximation	References
1.5PN comparable mass	Damour (gr-qc/0103018)
2PN comparable mass, BH induced quadrupole	Tanay+ (2012.06586)
One spin off, or both on, but to linear order, all PN orders	Wu & Xian (1004.4549)
Kerr geodesics (primary spin)	Carter (1968)
Spinning test particle in Kerr	Rüdiger (1981,1983), VW (1903.03651), Skoupý & VW (2411.16855)
Spinning “test black hole” in Kerr to quadrupole	Compère+ (2302.14549), Ramond (2402.02670)

Takeaway: Non-integrability may appear through >2PN, ~ $\nu S_1 S_2$ terms for BH binaries! (earlier for NSs)

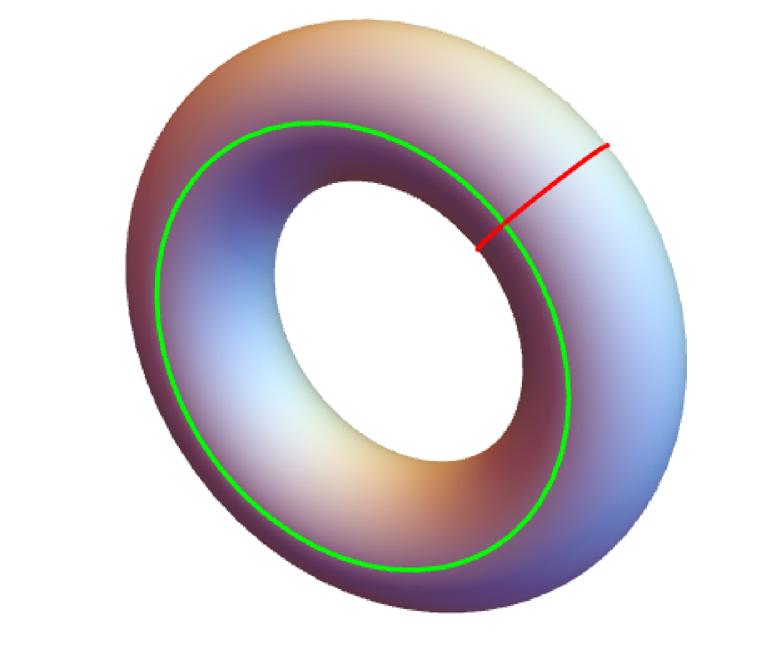
NEAT USE CASE: ACTIONS AS ROSETTA STONES FOR COMPACT BINARIES ACROSS THE PARAMETER SPACE

From VW, Skoupý, Stein, Tanay, 2411.09742

ACTIONS FOR SPINNING TEST PARTICLES

Take characteristic function fulfilling Hamilton-Jacobi equation from VW (1903.03651)

$$W = -E_{\mathrm{so}}(t - t_0) + L_{\mathrm{so}}(\phi - \phi_0) + (s_\parallel - s)(\psi - \psi_0) + \sum_{y=r,z} \int \pm \left(\sqrt{w_y'^2 + e_{0y} e^{\kappa}_{C;y} e_{D\kappa} \tilde{s}^{CD}} - \frac{1}{2} e^{\kappa}_{A;\mu} e_{B\kappa} \tilde{s}^{AB} \right) \mathrm{d}y$$



Compute actions over loops in phase space

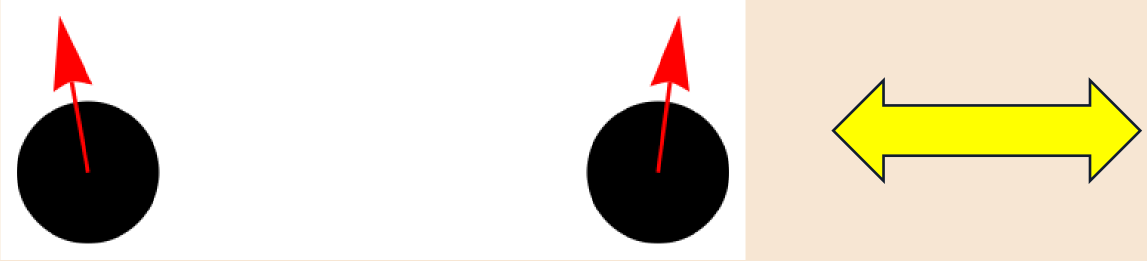
$$J_\gamma = \frac{1}{2\pi} \oint_\gamma \sum_q \pi_q \mathrm{d}q$$

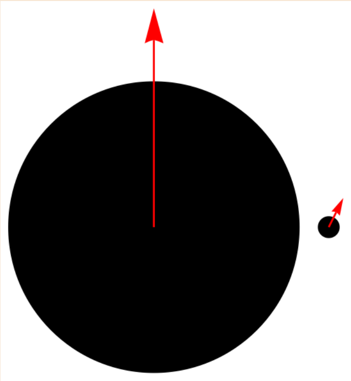
- Expand in spin, cure singularities
- Results in Legendre form in VW+ 2411.09742
- Also fundamental frequencies in closed form

MATCHING FROM PN TO TEST PARTICLE

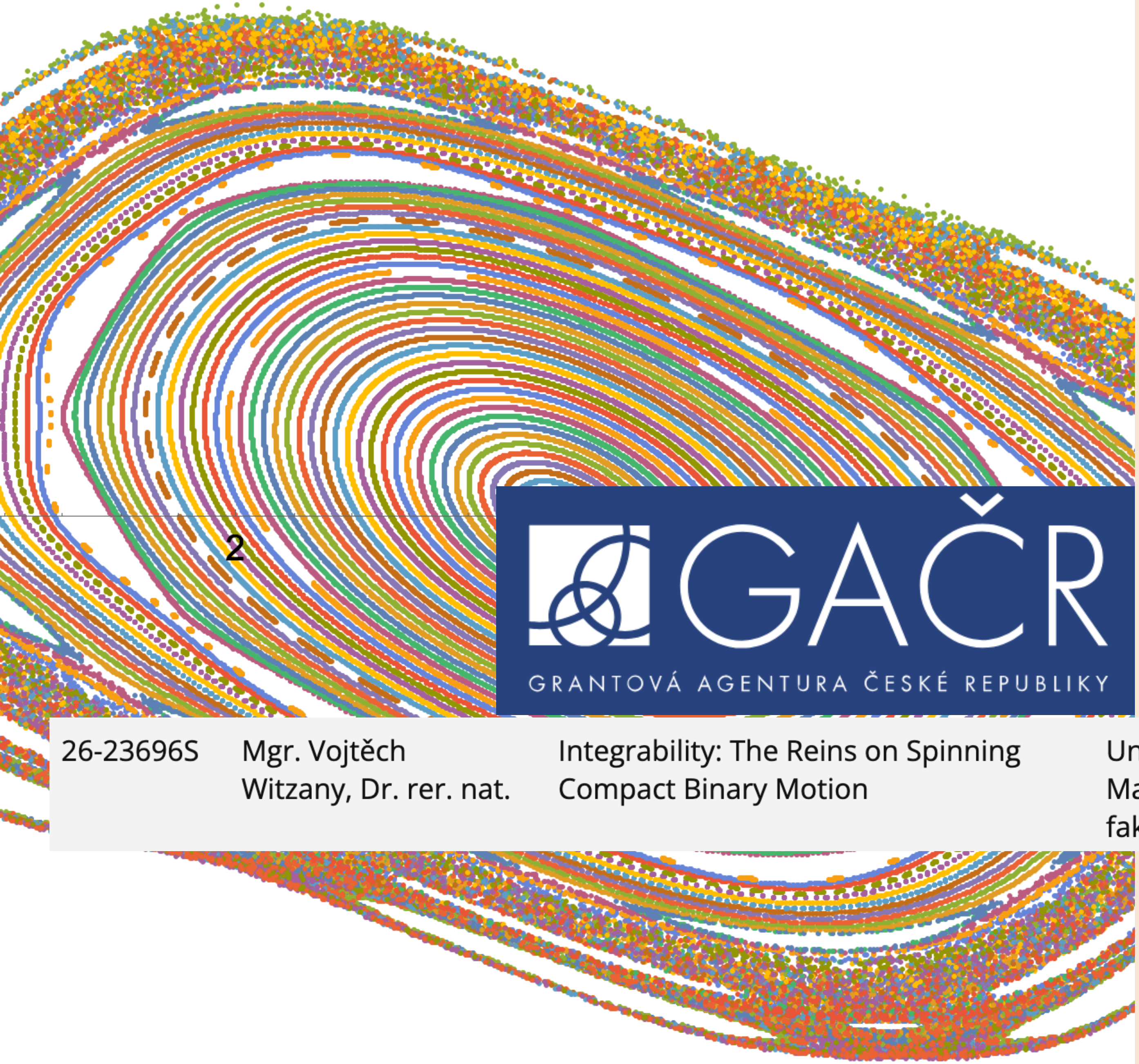
- Actions are coordinate-independent, only depend on homotopy class of loops
- Known in 1.5 PN limit from Tanay+ (2012.06586, 2110.15351)
- Subject only to $SL(\mathbb{Z},n)$ discrete lattice transforms
- The PN and test particle *continuous limits of the same system* (Used in the original EOB model by Buonanno & Damour, gr-qc/9811091)
- *Actions match in overlap up to $SL(\mathbb{Z},n)$, correspondence should hold everywhere*

$$
\begin{aligned}
I_r &= J_r\,,\\
I_L &= J_z + |J_\phi| - (J_\psi + s)\,,\\
I_{\Delta J} &= J_\phi\,,\\
I_5 &= J_\psi + s\,.
\end{aligned}
$$





(VW+ 2411.09742)



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SUMMARY

Integrability is practical and useful in more than one way

Integrability of spinning compact binaries to surprisingly high order, *not guaranteed by symmetry!*

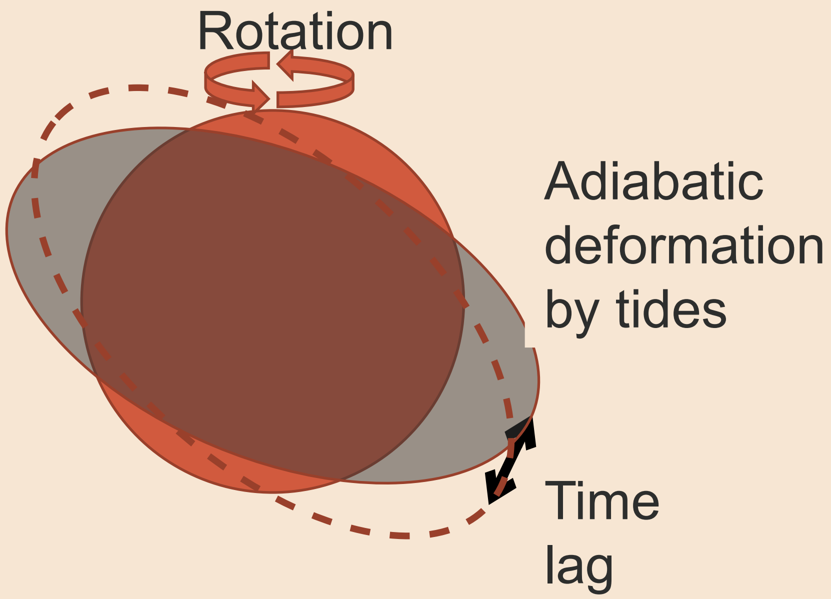
26-23696S	Mgr. Vojtěch Witzany, Dr. rer. nat.	Integrability: The Reins on Spinning Compact Binary Motion	Univerzita Karlova, Matematicko-fyzikální fakulta	3	OK2 – vědy o neživé přírodě

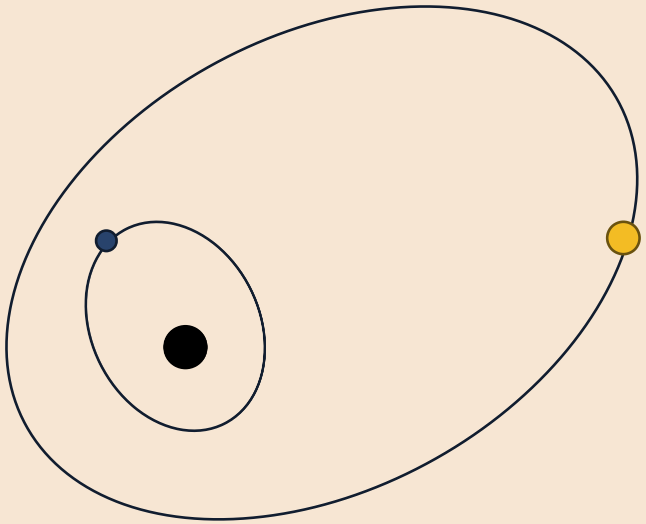
inherent to environmental, tidal, non-GR effects

BACKUP SLIDES

CONSERVATIVE???

- Approximations where only finite number of degrees of freedom are are needed to describe system and system is closed/autonomous
- Post-Newtonian dynamics to 2PN order on the level of equations of motion
- Higher orders with “standing-wave” boundary conditions up to 5PN (symmetric Green’s function)
- Standing-wave conditions also admissible for first order self-force
- Useful for two-timescale analysis





OTHER INTEGRABILITY BREAKING?

- Neutron stars, white dwarves have infinite internal degrees of freedom
- Once some internal degrees couple (*f*-modes etc.), this will break integrability at some point (e.g. Steinhoff+, 1608.01907)
- High order both in mass ratio and PN
- Other degrees of freedom are ”more stuff” in the system, such as a third body (Bonga+, 1905.00030)
- *Non-integrability is **generic,** integrability is special, any modification of gravity or environment will generically break it!*